Multivariable Integral Calculus

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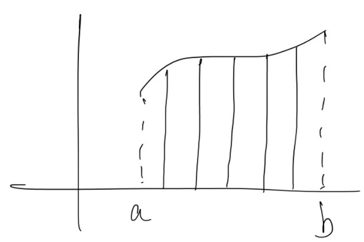
There are many resources out there to learn multivar calc.
This handout borrows heavily from Professor Leonard's playlist.
You could almost say it's a comprehensive study note from that playlist.
Have a look here!

1 Intro To Double Integrals

General idea in Calc 1:



Say we have a function f(x) which is a curve on the plane and we look at it in the interval $[a, b]$.
Take $[a, b]$, divide into n equal parts:

$$\frac{b-a}{n} = \Delta x \text{ (width of each partition)}$$

Find one point on each sub-interval:

$$x_k^{\cdot}$$

Find height at each $x_k^{\cdot}$:

$$f(x_k^{\cdot})$$

Find area of each rectangle:

$$f(x_k^{\cdot}) \cdot \Delta x$$

Add them up

$$A = \sum_{k=1}^{n} f(\dot{x_k}) \cdot \dot{x_k} \text{ (approx.)}$$

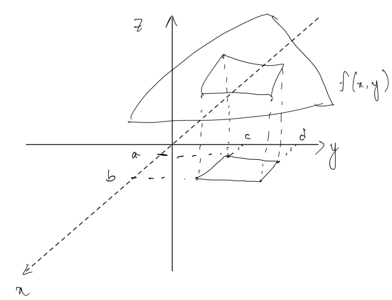
Take a limit for this now,

$$A = lim_{n \to \infty} \sum_{k=1}^{n} f(\dot{x_k}) \cdot \dot{x_k} \text{ (exact)}$$

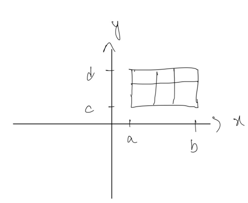
$$A = \int_a^b f(x)dx$$

What did we find? The area under the given curve. Now this idea extends to $3 - D$. In $3 - D$, we find out the volume under a given surface when one more independent variable is added.

IDEA 1:



One independent variable would give us a curve on the plane. Now, two independent variables give us a surface.
Let's look at a surface $f(x, y)$. If we look at an interval $[a, b]$ on the $x - axis$ and an interval $[c, d]$ on the $y - axis$, we can see a rectangular region being formed on the plane of the independent variables ($XY - plane$).



We now partition the rectangle into equal parts:

$$m : (\text{no. of x partitions})$$

$$n : (\text{no. of y partitions})$$

So according to it, in the figure above, we have, $m = 3$, $n = 2$.
Now similar to what we did in calc1 but a bit different, we pick $(x^{\cdot}_{ij}, y^{\cdot}_{ij})$ Then, we find the height:

$$f(x^{\cdot}_{ij}, y^{\cdot}_{ij})$$

By definition, we know,

$$V = S.A \times height$$

Now, if we cut x into m sub-intervals,

$$\frac{b-a}{m} = \Delta x \text{ (width of each rectangle)}$$

Similarly, if we cut y into n sub-intervals,

$$\frac{d-c}{m} = \Delta y \text{ (breadth of each rectangle)}$$

Then, area of each rectangle $= \Delta x \cdot \Delta y$

$$V = f(x^{\cdot}_{ij}, y^{\cdot}_{ij}) \cdot \Delta x \cdot \Delta y$$

We now add up the volumes of each rectangle,

$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^{\cdot}_{ij}, y^{\cdot}_{ij}) \cdot \Delta x \cdot \Delta y \text{ (approx.)}$$

Taking a limit for this now,

$$V = \lim_{(m,n)\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^{\cdot}_{ij}, y^{\cdot}_{ij}) \cdot \Delta x \cdot \Delta y \text{ (exact)}$$

$$V = \iint_R f(x, y)\, dx\, dy$$

where R is the region(for the figure above, we had a rectangular region).

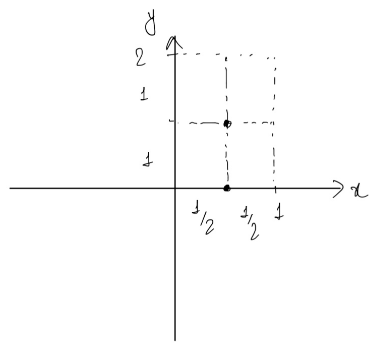
Let's look at an example to apply everything we've learnt so far.

Example

Find the volume under the surface $z = 8 - 2x^2 - y^2$ over the region

$$R = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2\}$$

for $m = 2$, $n = 2$.



Solution:
calculating width for each rectangle using the given fact that $0 \leq x \leq 1$ and $m = 2$.

$$\Delta x = \frac{1-0}{2} = \frac{1}{2}$$

similarly, calculating breadth for each rectangle

$$\Delta y = \frac{2-0}{2} = 1$$

$$\Delta A = \Delta x . \Delta y$$

$$\therefore A = \frac{1}{2} . 1 = \frac{1}{2}$$

Now,

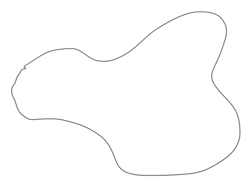
$$V \approx [f(\frac{1}{2}, 0) + f(\frac{1}{2}, 1) + f(1, 0) + f(1, 1)] \cdot \frac{1}{2}$$

$$\approx [\frac{15}{2} + \frac{13}{2} + 6 + 5] \cdot \frac{1}{2} = \frac{25}{2}$$

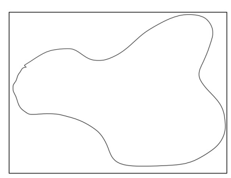
If we choose different sample points within each rectangle, the values of $f(x, y)$ will change, and therefore the total in the Riemann sum will also change. This is expected cause a Riemann sum is only an approximation, because we are replacing a continuous surface with a finite number of rectangular boxes. As we refine the partition (using more and smaller rectangles), these approximations become more accurate and eventually converge to the exact volume given by the double integral.

IDEA 2:

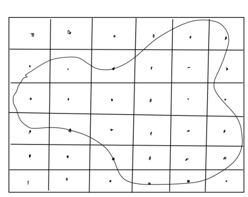
The region doesn't over which we want to find out the volume doesn't always have to be rectangular. Yes, it looks awkward.



However, if it's closed, it can be bound by a rectangle.



To just get the idea now, think of making partitions within the bounded rectangle like we did previously and then think of a point being in the center of the rectangle.



If the point lies within the boundary or inside of the region, then think that we consider that part for getting the area of the region. This might look a bit inaccurate(cause it is, and it's just an approximation), but if we now bring in the idea of *limit* and partition the rectangle that bounded the region into infinitely many rectangles, that would give us the area for the non-rectangular region.

2 Solving Double Integrals

For $f(x, y)$, the volume under the surface, over the region, R, is given by:

$$V = \iint_R f(x, y)\, dA$$

where $dA = dx.dy$ or $dA = dy.dx$ (yes)
So, for a rectangular region given by

$$R = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$$

$$V = \int_c^d \int_a^b f(x, y)\, dxdy \text{ or } \int_a^b \int_c^d f(x, y)\, dydx$$

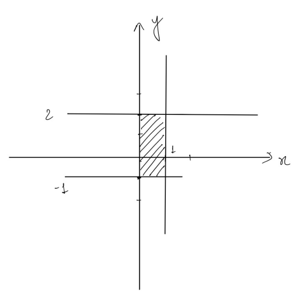
Just be cautious about the fact that you have to match up the intervals for x and y, and this idea that the order in which dy and dx are placed doesn't matter is called **Fubini's Theorem**.

So how do you choose the order when you're given the question? The first thing you should consider is making life easier for yourself. Just arrange it in a way that you think would be easier to solve for now. We'll look at different scenarios ahead.

Example

$$\iint_R x + y^2\ dA, \quad R : \{(x, y) \mid 0 \leq x \leq 1, -1 \leq y \leq 2\}$$

Solution:
For rectangular regions like this one, you don't really have to draw regions, but let's just do one for this question's sake. And you'll definitely need to draw figures for a lot of upcoming non-rectangular regions.



Let's just make sure the variable with respect to which we are about to integrate and the bounds are okay. Then, we just treat the other variable as a constant and integrate like we always did.

$$\begin{aligned}
\int_{-1}^{2}\int_{0}^{1}(x+y^2)\,dx\,dy &= \int_{-1}^{2}\left(\int_{0}^{1} x\,dx + \int_{0}^{1} y^2\,dx\right)dy \\
&= \int_{-1}^{2}\left(\frac{x^2}{2}\Big|_0^1 + y^2\cdot x\Big|_0^1\right)dy \\
&= \int_{-1}^{2}\left(\frac{1}{2}+y^2\right)dy \\
&= \frac{1}{2}(y)\Big|_{-1}^{2} + \frac{y^3}{3}\Big|_{-1}^{2} \\
&= \frac{3}{2}+3 \\
&= \frac{9}{2}
\end{aligned}$$

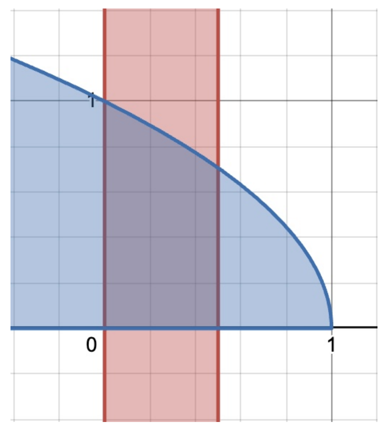
Just for this question, to show that Fubini's Theorem at work, let's do it with dy as the inner variable of integration.

$$\begin{aligned}
\int_{-1}^{2}\int_{0}^{1}(x+y^2)\,dx\,dy &= \int_{0}^{1}\int_{-1}^{2}(x+y^2)\,dy\,dx \\
&= \int_{0}^{1}\left(\int_{-1}^{2} x\,dy + \int_{-1}^{2} y^2\,dy\right)dx \\
&= \int_{0}^{1}\left(x(y)\Big|_{-1}^{2} + \frac{y^3}{3}\Big|_{-1}^{2}\right)dx \\
&= \int_{0}^{1}\left(x(2-(-1)) + \left(\frac{8}{3}-\frac{-1}{3}\right)\right)dx \\
&= \int_{0}^{1}(3x+3)\,dx \\
&= \left(\frac{3x^2}{2}+3x\right)\Big|_0^1 \\
&= \frac{3}{2}+3 \\
&= \frac{9}{2}
\end{aligned}$$

Example

$$\iint_R 2xy\,dA,\ \ R:\{(x,y)\mid 0\le x\le \frac{1}{2}\,,\,0\le y\le \sqrt{1-x}\}$$

Solution:



This is what it would look like if you plotted the graphs. You can see that the region is no longer rectangular.
Note: When setting up the inner integral from the graph, ensure that a single pair of boundary functions describes the region; avoid cases where you'd need to switch functions partway through the integration.
In this case, going with dx first you would have to switch boundary functions partway, but with dy you would have the same boundary functions.

$$\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-x}} 2xy\,dy\,dx$$

Note: While solving problems, you can also think of it as having the variable with constant values as boundaries as the outer variable of integration.

$$\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-x}} 2xy\,dy\,dx$$

$$=\int_0^{\frac{1}{2}}\left(\int_0^{\sqrt{1-x}} 2xy\,dy\right)dx$$

$$=\int_0^{\frac{1}{2}} 2x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x}}dx$$

$$=\int_0^{\frac{1}{2}} x\left(\sqrt{1-x}\right)^2 dx$$

$$=\int_0^{\frac{1}{2}} x(1-x)\,dx$$

$$= \int_0^{\frac{1}{2}} (x - x^2)\, dx$$
$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^{\frac{1}{2}}$$
$$= \frac{1}{8} - \frac{1}{24} = \frac{1}{12}$$

Fubini's Theorem for General Regions

For regions that are not rectangular, we can still apply Fubini's Theorem by expressing the region in a form where one variable has constant bounds and the other has bounds defined by curves as in the previous example. The key idea is to choose the order of integration so that the boundary functions do not switch within the region.

Example

Region Between $y = x^2$ and $y = 2x$. Consider the region bounded by the curves

$$y = x^2 \quad \text{and} \quad y = 2x$$

First, we find the points of intersection:

$$x^2 = 2x \quad \Rightarrow \quad x^2 - 2x = 0 \quad \Rightarrow \quad x(x-2) = 0,$$

so the curves intersect at

$$(x, y) = (0, 0) \text{ and } (2, 4)$$

To set up $\iint_R f(x,y)\, dA$, we describe the region vertically (in terms of y):

$$R = \{(x, y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$$

Thus the integral becomes

$$\iint_R f(x,y)\, dA = \int_0^2 \int_{x^2}^{2x} f(x,y)\, dy\, dx$$

Alternatively, we may reverse the order. Solving the curves for x:

$$x = \sqrt{y}, \qquad x = \frac{y}{2}$$

Since $x = \frac{y}{2}$ is left of $x = \sqrt{y}$ for $0 \le y \le 4$, the region can also be written as

$$R = \{(x, y) \mid 0 \le y \le 4,\ y/2 \le x \le \sqrt{y}\}$$

So reversing the order gives

$$\iint_R f(x,y)\, dA = \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\, dx\, dy$$

Both integrals compute the same area (or volume under a surface), illustrating Fubini's Theorem for general curved regions.
For video explanation you can start exactly by clicking **here**.

Example

Here's an example of an integral where, although either order of integration is possible, choosing the right setup makes life much easier.

$$\iint_R \frac{x}{1+xy} dA, \quad R: \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$$

If we set up the bounds, the integral looks like this. It shouldn't be the hardest part of solving this one at least.

$$\int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx$$

Treating x as a constant and integrating with respect to y, we look at the inner integral

$$\int_0^1 \frac{x}{1+xy}\, dy$$

Using u-substitution,

$$u = 1 + xy, \qquad du = x\, dy, \qquad dy = \frac{du}{x}$$

the limits change as follows: when $y = 0$, $u = 1$, and when $y = 1$, $u = 1 + x$

We now have,

$$\int_0^1 \frac{x}{1+xy}\, dy = \int_1^{1+x} \frac{x}{u} \cdot \frac{du}{x} = \int_1^{1+x} \frac{1}{u}\, du = \ln(1+x)$$

So it is now

$$\int_0^1 \ln(1+x)\, dx$$

Using integration by parts,

$$\int \ln(1+x)\, dx = (1+x)\ln(1+x) - x + C$$

Evaluating from 0 to 1, we obtain

$$\int_0^1 \ln(1+x)\, dx = \Big[(1+x)\ln(1+x) - x\Big]_0^1$$

Evaluating using the given bounds, we get,

$$\iint_R \frac{x}{1+xy}\, dA = 2\ln 2 - 1$$

Now imagine doing the same problem with dx as the inner variable of integration. Tough! Always try to make your life easier.

Exercise Problems

Try solving each of the following integrals using the techniques discussed so far. Choose the order of integration that makes life the easiest, and sketch the region whenever helpful.

P1

$$\iint_R \frac{ln(y)}{y}\,dA, \qquad R : \{(x,y) \mid 0 \leq x \leq \pi,\ e^{-2x} \leq y \leq e^{cos(x)}\}$$

$\frac{\pi}{4} - \frac{2}{3}\pi^3$

P2

$$\iint_R x^2 y\,dA, \qquad R : \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x^2\}$$

$\frac{1}{10}$

P3

$$\iint_R y e^{xy}\,dA, \qquad R : \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$$

$e - 2$

P4

$$\iint_R \frac{x}{1+y}\,dA, \qquad R : \{(x,y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 1\}$$

$2\ln 2$

P5

$$\iint_R \frac{1}{xy}\,dA, \qquad R : \{(x,y) \mid y \leq x \leq y^2,\ 1 \leq y \leq e\}$$

$\frac{1}{2}$

Example

3 Vector Calculus

3.1 Vector Fields

What is a vector field? Think of it as something

Cylindrical and Spherical Coordinates

They are going to save your lives btw.

Cylindrical Coordinates

They are just polar co-ordinates with a $z - component$.

$$Polar(r, \theta) \longrightarrow Cylindrical(r, \theta, z)$$

To get from cylindrical to rectangular co-ordinate system:

$$x = rcos\theta$$

$$y = rsin\theta$$

$$z = z$$

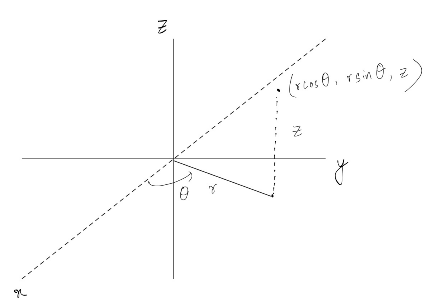
To get from rectangular to cylindrical co-ordinate system:

$$r^2 = x^2 + y^2$$

$$tan\theta = \frac{y}{x}$$

$$z = z$$

Note:
1) r is on the $xy - plane$. it is not the distance from the origin to the point, but the distance from the origin to the projection of the point on the $xy - plane$.
2) must match your quadrants and octants when converting points.



This figure should make things clearer. Let's take an example.

Example

Convert to rectangular co-ordinate system. Given:

$$Cylindrical : (3, -\frac{\pi}{6}, 2)$$

Solution:
We have,

$$r = 3,\ \theta = \frac{-\pi}{6},\ z = 2$$

Looking at the angle θ, we can tell that it's in quadrant 4 and since it has a positive $z-$value it's in octant 4.

$$x = 3cos(\frac{-\pi}{6}) = 3cos(\frac{\pi}{6})\{\because cos(-\theta) = cos(\theta)\} = \frac{3\sqrt{3}}{2}$$
$$y = 3sin(\frac{-\pi}{6}) = \frac{-3}{2}$$
$$z = 2$$
$$\therefore \text{in rectangular coordinates} \longrightarrow (\frac{3\sqrt{3}}{2}, \frac{-3}{2}, 2)$$

Example

Convert to cylindrical co-ordinate system. Given:

$$Rect : (\sqrt{2}, -\sqrt{2}, 4)$$

Solution:
We have,

$$x = \sqrt{2},\ y = -\sqrt{2},\ z = 4$$

since $(x, y) \rightarrow (+, -)$, it's in quadrant 4, and since the $z-$value is positive too, it's in the octant 4.

$$r^2 = x^2 + y^2 = (\sqrt{2})^2 + (-\sqrt{2})^2$$
$$r^2 = 4$$
$$r = 2 \text{ or } -2$$
$$tan\theta = \frac{y}{x},\ tan\theta = \frac{-\sqrt{2}}{\sqrt{2}}$$
$$tan\theta = -1,\ \theta = tan^{-1}(1)$$
$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

The point was in quadrant 4, therefore we pick $\frac{7\pi}{4}$ for the $+ve\ r$, and $\frac{3\pi}{4}$ for the $-ve\ r$.

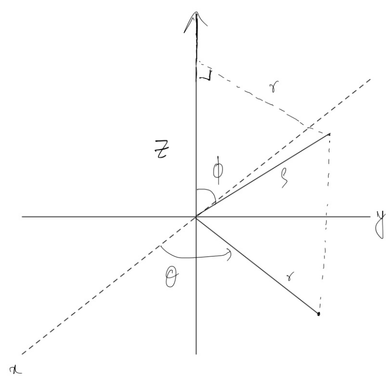
$$\therefore Cylindrical : (2, \frac{7\pi}{4}, 4), (-2, \frac{3\pi}{4}, 4)$$

Note:
Use $(+)ve\ r$ with θ from proper quadrant.
Use $(-)ve\ r$ with the other θ you obtain.

Spherical Coordinates

$$Spherical : (\rho, \theta, \phi)$$

◦ Uses the distance ρ from the origin to the point. $\rho \geq 0$
◦ θ, just like cylindrical coordinate system. $0 \leq \theta \leq 2\pi$
◦ ϕ, angle from $(+)ve\ z - axis$, **Note:** $0 \leq \phi \leq \pi$



This figure should make things clearer.
To get from spherical to rectangular co-ordinate system:
We know,

$$x = rcos\theta,\ y = rsin\theta$$

but look at the figure above and something we can say, using trigonometric ratios, is

$$r = \rho sin\phi$$

$$\therefore x = \rho sin\phi cos\theta,\ y = \rho sin\phi sin\theta$$

likewise,

$$z = \rho cos\phi$$

To get from rectangular to spherical co-ordinate system:

$$\rho^2 = x^2 + y^2 + z^2$$

$$tan\theta = \frac{y}{x}$$

$$cos\phi = \frac{z}{\rho}$$

Example

Convert to rectangular co-ordinate system. Given:

$$Spherical : (3, \frac{\pi}{4}, \frac{3\pi}{4})$$

Solution:
We have,

$$\rho = 3,\ \theta = \frac{\pi}{4},\ \phi = \frac{3\pi}{4}$$

$$x = \rho sin\phi cos\theta = 3sin(\frac{3\pi}{4})cos(\frac{\pi}{4}) = \frac{3}{2}$$

$$y = \rho sin\phi sin\theta = 3sin(\frac{3\pi}{4})sin(\frac{\pi}{4}) = \frac{3}{2}$$

$$z = \rho cos\phi = 3cos(\frac{3\pi}{4}) = \frac{-3\sqrt{2}}{2}$$

$$\therefore \text{in rectangular coordinates} \longrightarrow (\frac{3}{2}, \frac{3}{2}, \frac{-3\sqrt{2}}{2})$$

Example

Convert to spherical co-ordinate system. Given:

$$Rect : (-2, 2\sqrt{3}, 4)$$

Solution:
We have,

$$x = -2,\ y = 2\sqrt{3},\ z = 4$$

since $(x, y) \rightarrow (-, +)$, it's in quadrant 2, and since the $z-$value is positive too, it's in the octant 2.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-2)^2 + (2\sqrt{3})^2 + (4)^2}$$

$$\therefore \rho = \sqrt{4 + 12 + 16} = \sqrt{32} = 4\sqrt{2}$$

$$tan\theta = \frac{2\sqrt{3}}{-2}$$

$$tan\theta = -\sqrt{3}$$

$$\therefore \theta = tan^{-1}(\sqrt{-3}) = \frac{2\pi}{3}, \frac{5\pi}{3}$$

since the given point is in $2nd$ quadrant, we take $\theta = \frac{2\pi}{3}$

$$\phi = cos^{-1}(\frac{z}{\rho}) = cos^{-1}(\frac{4}{4\sqrt{2}}) = \frac{\pi}{4}$$

$$\therefore Spherical : (4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4})$$

Let's now look at some examples where we take cylindrical coordinates to spherical and vice-versa.

Note:
1) θ will always be common for cylindrical and spherical co-ordinate systems.
2 Always calculate ρ first when converting to spherical because you'll need it to find ϕ.

Example

Convert to spherical co-ordinate system. Given:

$$Cylindrical : (4, \frac{\pi}{3}, -4)$$

Solution:
We have,

$$r = 4,\ \theta = \frac{\pi}{3},\ z = -4$$

Now, we know,

$$\rho^2 = x^2 + y^2 + z^2$$
$$\rho^2 = r^2 + z^2\ \{\because x^2 + y^2 = r^2\}$$
$$\rho = \sqrt{4^2 + (-4)^2}$$
$$\therefore \rho = 4\sqrt{2}$$

We already know what θ is. Lastly,

$$cos\phi = \frac{z}{\rho}$$
$$cos\phi = \frac{-4}{4\sqrt{2}}$$
$$cos\phi = \frac{-1}{\sqrt{2}}$$
$$\therefore \phi = \frac{3\pi}{4}$$

The nice thing about ϕ as you must have noticed, is that the interval goes between $0 \leq \phi \leq \pi$ so you only obtain one value for *cosine*.

$$\therefore Spherical : (4\sqrt{2}, \frac{\pi}{3}, \frac{3\pi}{4})$$

Example

Convert to cylindrical co-ordinate system. Given:

$$Spherical : (5, \frac{\pi}{4}, \frac{3\pi}{4})$$

Solution:
We have,

$$\rho = 5,\ \theta = \frac{\pi}{4},\ \phi = \frac{3\pi}{4}$$

We know,

$$r = \rho sin\phi,\ z = \rho cos\phi$$

$$r = 5sin(\frac{3\pi}{4}),\ z = 5cos(\frac{3\pi}{4})$$

$$\therefore r = \frac{5}{\sqrt{2}},\ z = \frac{-5}{\sqrt{2}}$$

Hence,

$$Cylindrical : (\frac{5}{\sqrt{2}}, \frac{\pi}{4}, \frac{-5}{\sqrt{2}})$$

We had a look at straightforward examples and conversion from one co-ordinate system to the other. Let's now look at some more examples of how we can use these co-ordinate systems.

DEFINITIONS

Fubini's Theorem:

If $f(x, y)$ is a continuous function on a rectangle $R = [a, b] \times [c, d]$, then the double integral

$$\iint_R f(x, y)\, dA$$

is equal to the iterated integral

$$\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$$

and also to the iterated integral

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

In both cases, we evaluate the inner integral using the fundamental theorem of calculus, treating the other variable as a constant. (*source*)

Other Resources For Studying